Summary

Gears scale and transmit force/torque, power, and energy across machines and mechanisms. Manufacturers produce various types of gears (e.g., helical, spur, herringbone, worm, rack and pinion, bevel and hypoid) for different applications and design tradeoffs (e.g., cost, efficiency, vibration/noise, angled-drives, etc.)

Gear teeth: do they break? Loads, geometry, and material.

Three factors for designing gear teeth are:

1. ***Contact stress:*** As the photoelastic image to the right shows, there is high stress around the gear's contact point due to the contact force $\vec{\mathbf{F}}$ near the tip of the gear tooth.

2. ***Bending stress:*** The photoelastic image also shows high stress around the base of the tooth due to bending caused by $\vec{\mathbf{F}}$. The bending causes the left-side base of the tooth to be in tension whereas the right-side base is in compression.

3. ***Fatigue:*** Gears rarely fail due to a single application of force $\vec{\mathbf{F}}$. Problems are exacerbated by cyclical loading. Each revolution of the gear causes loading and unloading of each tooth. To account for fatigue, we estimate actual gear stresses and compare them to ***maximum allowable lifetime stresses***. The process is not hard, but requires iteration (e.g., with MATLAB® or Python or a spreadsheet).

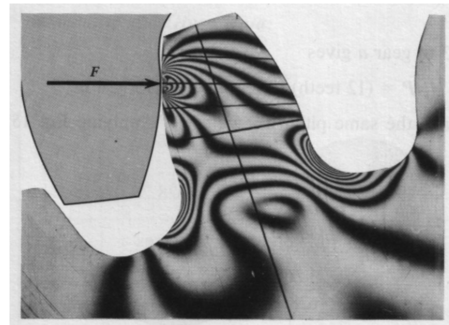


Figure 15.19 Photoelastic pattern of stresses in a spur gear tooth. (From T. J. Dolan and E. L. Broghammer, A Study of Stresses in Gear Tooth Fillets, Proc. 14th Eastern Photoelasticity Conf., PE December 1941.)

Experiment with fatigue:
Break a paper-clip by cyclical loading.

Example of cyclical loading in an automobile: With average driving/engine speed of 2000 rpm, the input gear of a transmission has

$$\frac{2000 \text{ rotations}}{\text{min}} * \frac{60 \text{ min}}{\text{hour}} * \frac{10 \text{ hours}}{\text{week}} * \frac{50 \text{ weeks}}{\text{year}} * 10 \text{ years} = \mathbf{6 \,x\, 10^8} \text{ cycles}$$

This section covers how to design gears to hold up over time. The basics derive from beam theory and contact stress theory. Both Renaissance gear teeth (shown right) and modern gear teeth with an involute profile (shown above-right) can be analyzed as cantilever beams. Note that the contact force $\vec{\mathbf{F}}$, which is along the ***line of action*** between the two gears, has both a tangential and radial component.

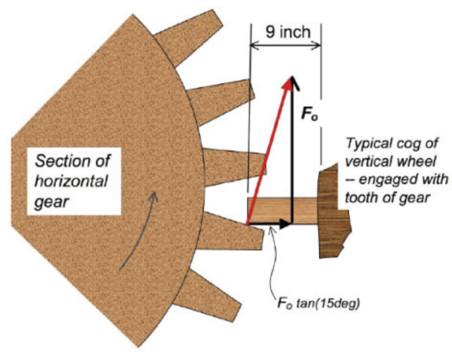


Figure 1. Renaissance gear with the usual normal and tangential forces (per Assignment 1).

1. We start by computing desired speed ratios (e.g., to get a desired output torque or speed).

2. Then we pick some standard gears with appropriate dimensions (experience helps here).

3. Next we compute gear stresses, including some empirical conversion and service factors that we can apply from tables and charts for our application.

4. We compare the estimated stresses to the allowable stresses and, if necessary, we iterate, with new gear choices (bigger or smaller, depending on the result).

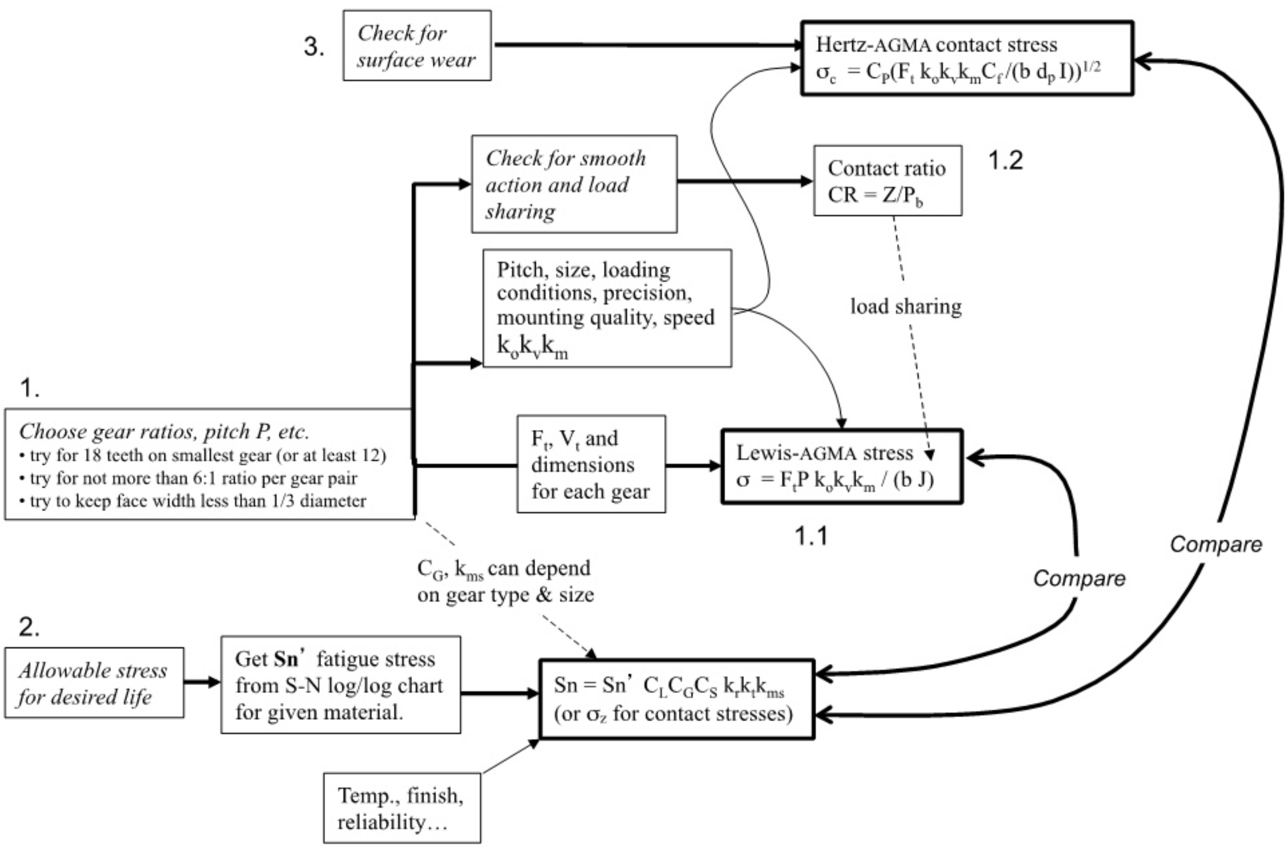
The diagram below shows the basic flow of calculations. The lines in bold are the main paths:

(a). Choose some gears; get dimensions; estimate the main bending stresses (Lewis formula).
At this time, also check the contact ratio (for smooth action and load sharing).

(b). Compare **(a)** with the allowable lifetime stresses in bending. Choose new gears if necessary.

(c). Also check the contact stresses and surface wear. Compare these against the allowable lifetime contact stress σ_z. Choose new gear dimensions or a harder material, if necessary.

As the flow chart shows, there are various secondary considerations (e.g. speeds, vibration, manufacturing quality) that affect calculations and require iteration. There are also many conversion factors which is why a spreadsheet or computer program is helpful.



1. Choose gear ratios, initial estimate of pitch P etc.

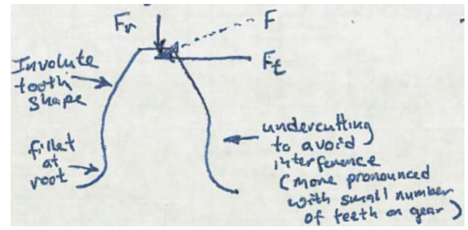
Based on your desired speed and torque ratios, you have an idea of the relative diameters of the two gears in a pair. At this point, take a look in a gear catalog. Recall Richard Feynman's advice[1].
Most catalogs have a chart or calculator to get an idea of the gear size. For example:

Boston Gear Catalog "theory"	http://www.bostongear.com/products/open/theory.html
Pic:	http://pic-designcatalog.com/cgi-bin/lansaweb?srve=PRDLST+F(LW3CATURL)=SPUR_IN

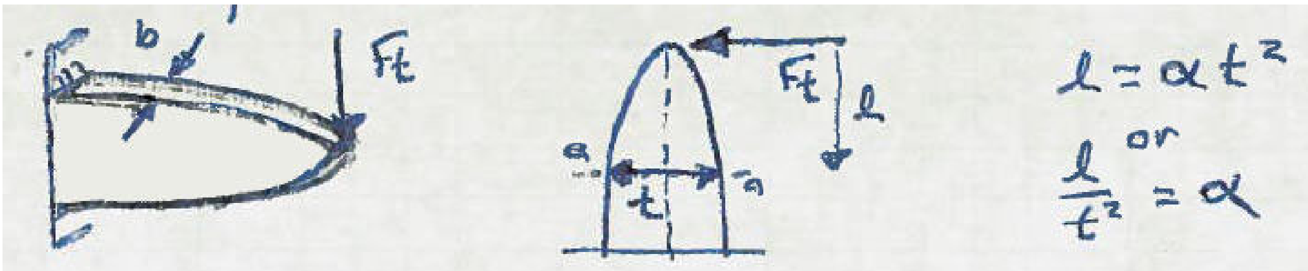
[1] http://bdml.stanford.edu/Main/FeynmanGears

2. Accounting for bending stress (Lewis/AGMA)

The tangential and radial stress produce stress in the gear tooth. The most significant of these is bending stress caused by the tangential force F_t. As shown right, a gear tooth has an interesting shape, which raises the question: "***where is stress highest?***"



In 1892, Wilfred Lewis presented an analysis that is still used as the basic principle for gear stress analysis. He started with the observation that a gear resembles a cantilever beam. He then used the result that a cantilever beam with a parabolic shape has equal strength when loaded at its tip. This idea dates to Galileo Galilee work in 1638 (*Discourses and Mathematical Demonstrations Relating to Two New Sciences*).



To see why, consider cross-section aa of a cantilever beam that is located a distance l from the beam's tip.

From beam-bending theory (Euler), the maximum bending stress σ is calculated as follows. This shows σ is **constant** for any arbitrary cross-section aa.

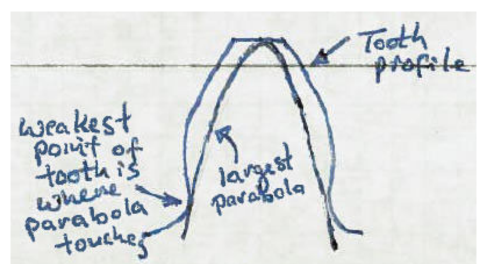
$$\sigma = \frac{M\,c}{I} = \frac{(F_t * l)\,\frac{t}{2}}{\frac{b\,t^3}{12}} = \frac{6\,F_t\,l}{b\,t^2} = 6\,F_t\,\frac{\alpha}{b}$$

F_t	Tangential force at tip of gear tooth (computed using known torque and pitch radius).
M	Moment $= F_t * l$ due to F_t at section aa.
c	Distance from beam's neutral axis ($c = t/2$).
b	Face width of gear tooth.
t	Thickness of gear tooth.

This shows that for a parabolic shape, the maximum bending stress σ does **not** depend on the distance l from the tip (Galileo actually proposed this based on ratios of volumes and not as a bending stress calculation).

Lewis used this idea to come up with strength factors for gear teeth graphically. Using large drawings of tooth profiles, he found the largest parabola that would fit and determined the strength accordingly.

This gave a geometrical factor in the stress analysis that was independent of ***pitch*** but dependent on the design pressure angle and the number of teeth (and the type of gear).



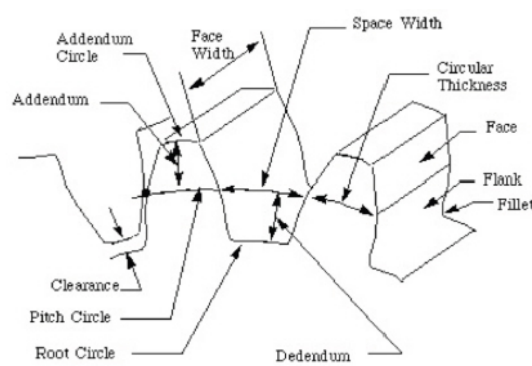
Lewis' initial calculations assumed F_T was applied at the tip since he argued that gears could not be precisely manufactured to assume a second tooth would help share the load (with modern manufacturing, this is no longer the case). Also, the stress concentration at the root of the tooth from the fillet was recognized as a contributor to the stress. The American Gear Manufacturer's Association (AGMA) publishes tables for stress analyses, tabulating geometry factors that combine Lewis's initial ideas with these two additions (load sharing and fillet/stress concentration).

The cantilever beam stress σ is written as:

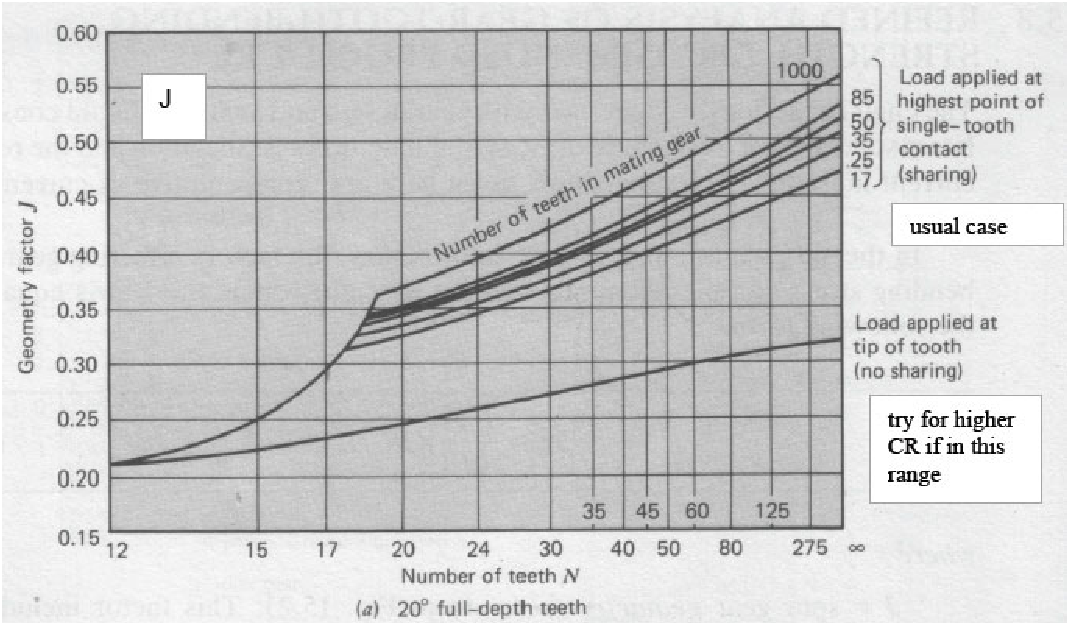
$$\sigma = \frac{M\,c}{I} = \frac{(F_t * l)\,c}{I} = \frac{F_t\,P}{b\,J} \qquad \text{Lewis formula}$$

P	***Diametral pitch*** (see figure to right)
J	***Lewis geometry factor*** depends on number of teeth of both gears in mating pair and on the ***Contact Ratio***. J is tabulated for various families of gears (next page).

Collectively, P, b, J account for l, c, I as $\frac{P}{b\,J} = \frac{l\,c}{I}$.



A worst case loading condition has F_t at the highest point of the tooth when only one tooth in contact. However, the geometry factor J depends upon the number of teeth in the gear. For precision gears, the number of teeth in the mating (other) gear is also important. This is because gears with a greater number of teeth tend to have more teeth in contact (on average).



Lewis factor J for 20° pressure angle gears. Worst case if l is full tooth length (F_t at tip).

AGMA uses a modified version of the Lewis formula, adjusting for things like speed (higher speeds produce larger dynamic loads), mounting rigidity, and shock and overload conditions.

Modified Lewis formula

$$\boxed{\sigma = \frac{F_t\,P}{b\,J}\,K_m\,K_o\,K_v}$$

K_m ***Mounting factor***: If gears are poorly mounted, load will be unevenly shared between teeth or across the face-width of the tooth. K_m depends on face-width and mounting accuracy.

TABLE 15.2
Mounting Correction Factor K_m

Characteristics of Support	Face Width (in.) 0 to 2	6	9	16 up
Accurate mountings, small bearing clearances, minimum deflection, precision gears	1.3	1.4	1.5	1.8
Less rigid mountings, less accurate gears, contact across the full face	1.6	1.7	1.8	2.2
Accuracy and mounting such that less than full-face contact exists	Over 2.2			

K_o ***Overload factor***: Shock loading can dislodge pinned dislocations and seriously shorten the fatigue life. Thus if the gears are subject to shock loads, this should be taken into account.

TABLE 15.1
Overload Correction Factor K_o

Source of Power	Driven Machinery: Uniform	Moderate Shock	Heavy Shock
Uniform	1.00	1.25	1.75
Light shock	1.25	1.50	2.00
Medium shock	1.50	1.75	2.25

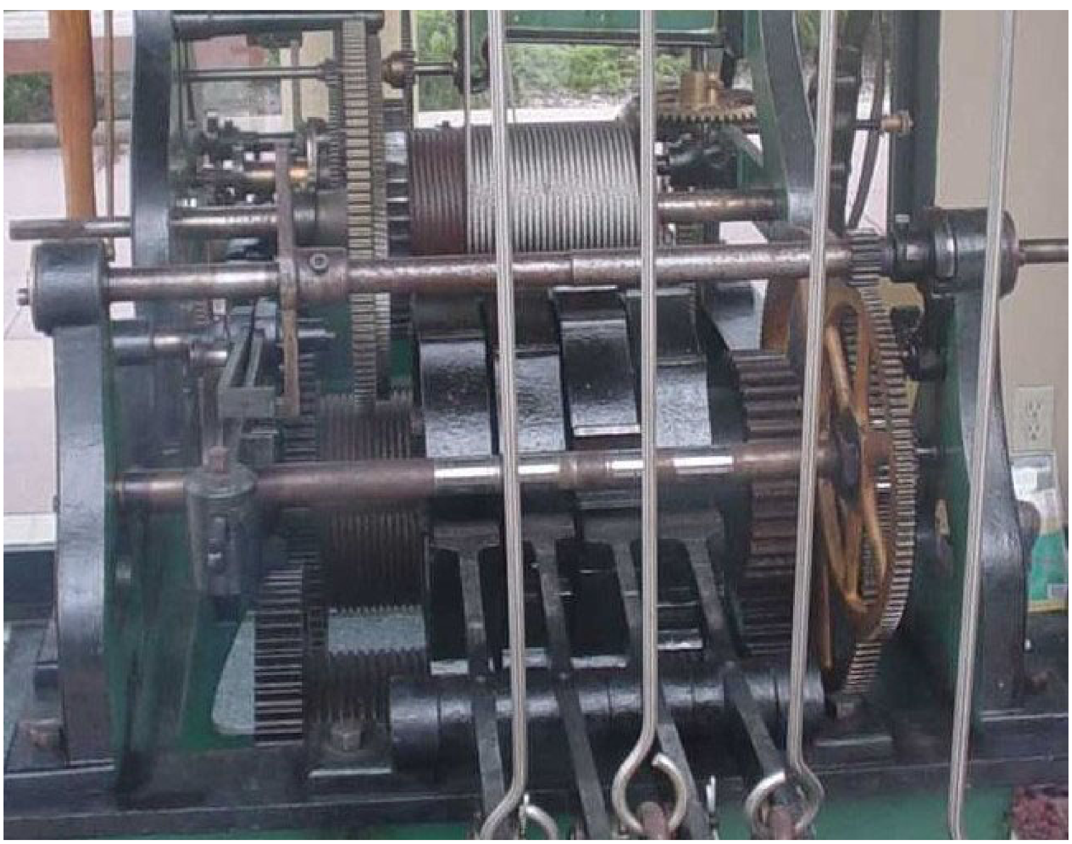
K_v ***Velocity factor***: K_v is based on the velocity $v = r_p\,\omega$ (r_p is the radius of the pitch circle) of the gear teeth along the tangent line where the gear meshes with its mating gear. There are higher loads when gear teeth impact. The variation in load on impact depends upon the precision of the gear teeth – with higher precision teeth requiring a factor between 1 and 1.5 (depending upon velocity v).

AGMA formulas: $K_v = \left(\dfrac{A + \sqrt{v}}{A}\right)^B$ where v is in $\dfrac{\text{ft}}{\text{min}}$ or $K_v = \left(\dfrac{A + \sqrt{200\,v}}{A}\right)^B$ where v is in $\dfrac{\text{m}}{\text{s}}$

$A = 50 + 56\,(1 - B)$ $B = 0.25\,(12 - Q_v)^{2/3}$ If $Q_v > 12$, let $B = 0$.

Q_v is the AGMA quality factor found in a catalog such as AGMA 10 or AGMA 12.

Example from Juvinill and Marshek for high precision gears: $K_v \approx \sqrt{\frac{78 + \sqrt{v}}{78}}$ where v is in $\frac{\text{ft}}{\text{min}}$



3. Compute allowable bending and contact lifetime stresses

For steel alloys commonly used in gears, etc. the failure trend is as follows. The heavy black line is regarded as a "safe" value for steel in bending. Su is the ***ultimate tensile stress***, $S_n^{'}$ (labeled simply as S in the ratio for the vertical axis in the figure below) is the ***allowable bending stress*** for a given number of cycles. Thus, if a steel has an ultimate stress of $Su = 200$ ksi, the allowable stress $S_n^{'}$ for 10^6 cycles is about $\frac{1}{2}$ of that.

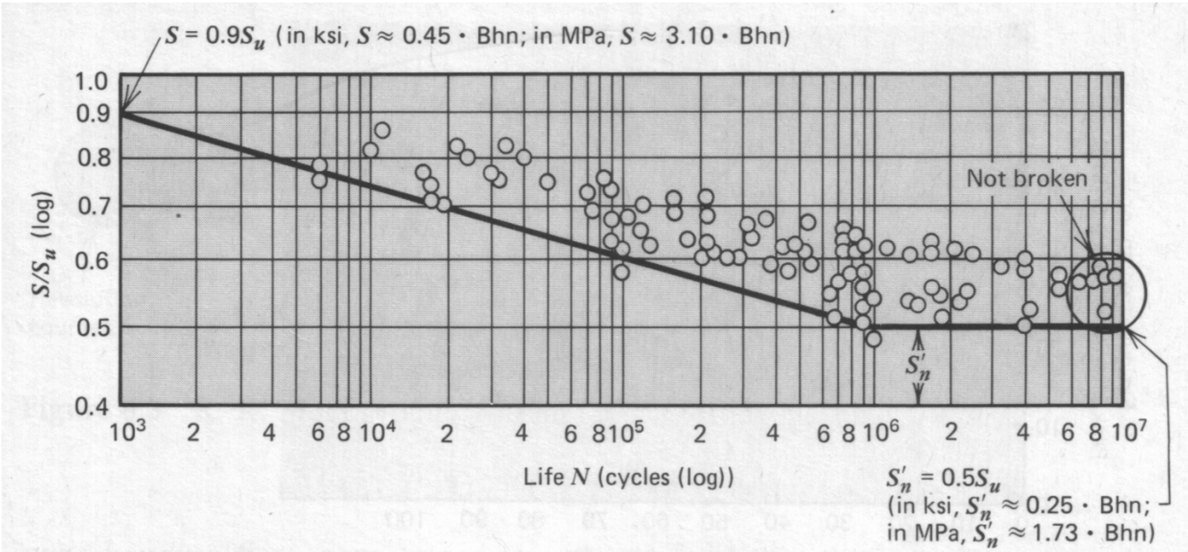


Figure 8.5 Generalized S–N curve for wrought steel with superimposed data points

$S_n^{'}$ from Juvinall and Marshek: Moore bending test data and conservative linear fit on log/log plot. ($\frac{S_n^{'}}{Su}$ versus cycles).

Finally, we adjust by various factors for ***allowable lifetime stress*** S_n.

$$S_n = S_n^{'} \; C_L \; C_G \; C_S \; k_t \; k_{ms} \; k_r$$

$S_n^{'}$ is the endurance limit from the data in J&M (previous figure).

C_L is a factor that accounts for the type of loading. $C_L = 1.0$ for bending loads (such as gear teeth experience).

C_G is a gradient factor that accounts for the fact that larger features have more places where cracks can start. $C_G = 1.0$ for gears of pitch $P \geq 5$; $C_G = 0.85$ for $P < 5$.

C_S accounts for the effects of surface finish and treatment. Smooth is better as seen right (in the figure).

k_t accounts for temperature. If $T < 160°$F, $k_t = 1$, else $k_t = \frac{620}{460+T}$.

k_{ms} accounts for mean stress. If stress is fully reversed, $k_{ms} = 1.0$. For non-idler gears, stress is in one direction so $k_{ms} = 1.4$.

k_r accounts for statistical distribution and desired reliability

TABLE 15.3
Reliability Correction Factor k_r, from Fig. 6.19 with Assumed Standard Deviation of 8 Percent

Reliability (%)	50	90	99	99.9	99.99	99.999
Factor k_r	1.000	0.897	0.814	0.753	0.702	0.659

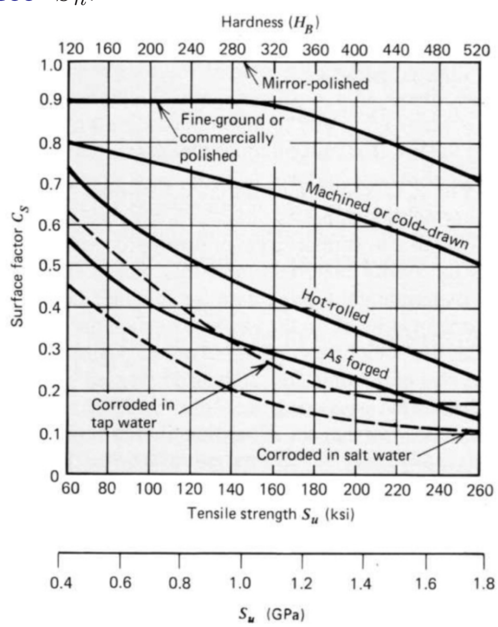
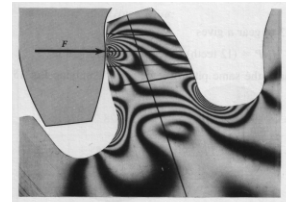


Fig 8.13 from J&M: Surface Cs factors for various types of steel. "Machined or cold-drawn" is appropriate for many commercial gears.

The ***allowable lifetime stress*** S_n must be compared against the computed stress in the gear teeth, with any additional safety factor applied as desired.

4. Check for contact stress and compare against allowable contact stress



As seen in the photoelastic image, gears are subject to wear due to contact stresses on their surfaces. The same is true for ball bearings, rollers, etc. The contacting faces are subject to Hertz contact stresses, which can produce failures such as pitting and erosion. Like bending stresses, contact stresses must be compared against corresponding allowable lifetime contact stresses.

Contact stress is computed as a variation of the classic Hertz contact stresses for spheres and cylinders. We approximate the curved gear tooth of each gear locally as a cylindrical surface of radius r.

$$\text{Hertz contact stress:}\quad \sigma_c \;=\; \underbrace{\sqrt{\frac{1}{\pi\,\frac{(1-\nu_1^2)}{E_1} + \pi\,\frac{(1-\nu_2^2)}{E_2}}}}_{C_p\ \ \text{(material property only)}}\;\sqrt{\frac{K_v\,F_t}{F\,\cos(\phi)}}\;\underbrace{\sqrt{\frac{1}{r_1}+\frac{1}{r_2}}}_{\text{(geometry)}}$$

$$\text{Written in AGMA form:}\quad \sigma_c \;=\; C_p\sqrt{\frac{F_t\,K_o\,K_v\,K_s\,K_m\,C_f}{dp\;b\;I}}$$

Rewriting this result in terms of standard gear parameters, we get the Hertz-AGMA contact stress:

$$\sigma_c \;=\; C_p\sqrt{\frac{F_t\,K_o\,K_v\,K_m\,C_f}{dp\;b\;I}} \qquad I \;=\; \frac{1}{2}\,\sin(\phi)\,\cos(\phi)\,\frac{R}{R+1} \qquad \left(\text{where}\;\; R=\frac{d_{\text{gear}}}{d_{\text{pinion}}}\;\; \text{using the pitch radii of each.}\right)$$

where we have assumed $C_f = K_s = 1$ (same materials and finish for both gears).

C_p is an elastic constant (see Tables 15.4a, 15.4b on next page) that depends on the Young's modulus of both the pinion and the gear it mates with.

d_p is pinion pitch diameter – diameter of smaller of the two meshing gears.

K_s size factor intended to account for material non-uniformity due to tooth size, material etc. (use $K_s = 1$ unless otherwise directed).

C_f relative hardness factor - Use 1 if both gears are the same material. Or look it up (e.g. Marks' Standard Handbook) for other cases.

Note: We do this analysis for the pinion (the smaller of the two gears) which has the higher cycles and stresses. If this pinion is OK, the other gear will be fine if it is made of the same material.

TABLE 15.4a
Values of Elastic Coefficient C_p for Spur Gears, in $\sqrt{\text{psi}}$ (Values Rounded Off)

Pinion Material (ν = 0.30 in All Cases)	Gear Material: Steel	Cast Iron	Aluminum Bronze	Tin Bronze
Steel, E = 30,000 ksi	2300	2000	1950	1900
Cast iron, E = 19,000 ksi	2000	1800	1800	1750
Aluminum bronze, E = 17,500 ksi	1950	1800	1750	1700
Tin bronze, E = 16,000 ksi	1900	1750	1700	1650

TABLE 15.4b
Values of Elastic Coefficient C_p for Spur Gears, in $\sqrt{\text{MPa}}$ (Values Converted from Table 15.4a)

Pinion Material (ν = 0.30 in All Cases)	Gear Material: Steel	Cast Iron	Aluminum Bronze	Tin Bronze
Steel, E = 207 GPa	191	166	162	158
Cast iron, E = 131 GPa	166	149	149	145
Aluminum bronze, E = 121 GPa	162	149	145	141
Tin bronze, E = 110 GPa	158	145	141	137

As the figure below shows (from Juvinall & Marshek), the allowable lifetime contact stresses σ_z are considerably higher than the allowable lifetime bending stresses. More specifically, for less than 10^7 cycles, the contact stresses can be rather high compared to the ultimate tensile stress in bending. However, for much higher cycles ($> 10^9$) they are comparable. Notice also that small, polished and hardened elements like ball bearings can sustain even higher lifetime contact stress than gear teeth.

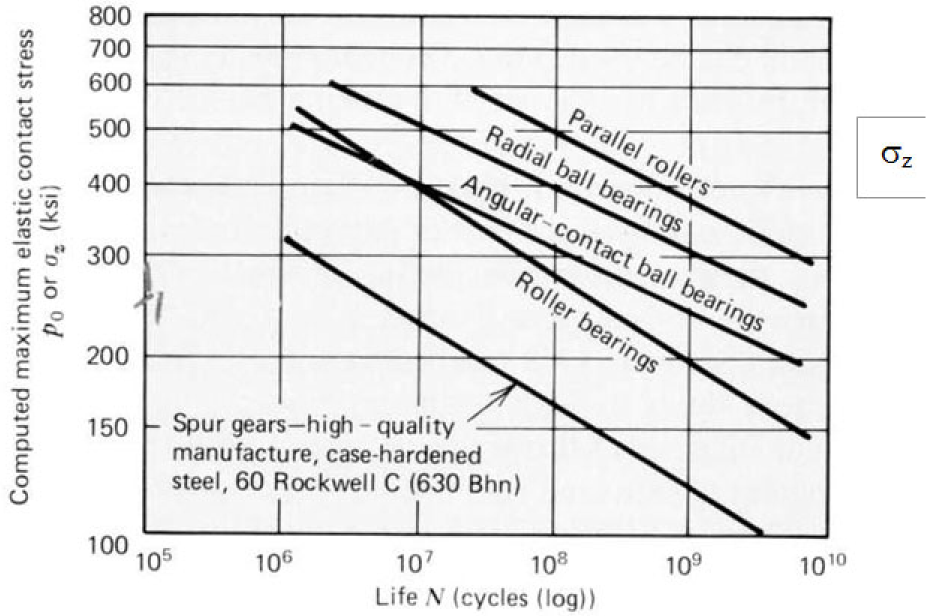


Figure 9.21 Average S–N curves for contact stresses—rollers, bearings, and spur gears, 10 percent failure probability [7].