See also: https://www.youtube.com/watch?v=4fMRIrNLB58
(in Mechanisms and Linkages module in Canvas)

4-bar Linkage position analysis

Terminology

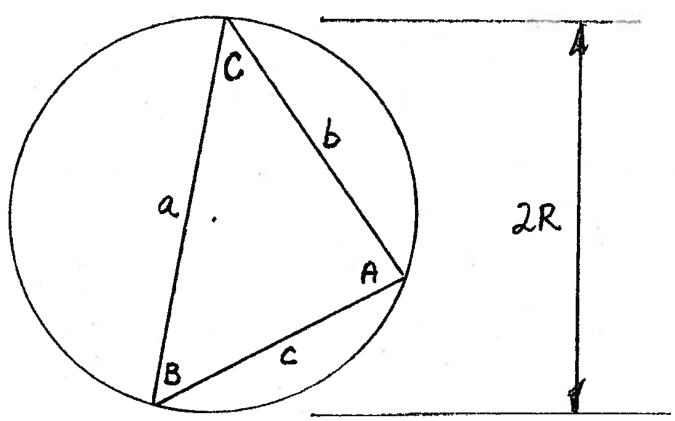
Vectors in a plane can be represented in several equivalent ways, using an (x, y) or (i, j) coordinate system:

$\mathbf{R} = [r_x, r_y]^t$, a column vector in Cartesian [x, y] coordinates such that $R = |\mathbf{R}| = \mathrm{sqrt}(r_x^2 + r_y^2)$
$\mathbf{R} = R\sin\theta\, i + R\cos\theta\, j$ in terms of angle, θ, and magnitude, R.
$\mathbf{R} = Re^{i\theta}$ in complex exponential notation

You may encounter any of these in kinematics textbooks.

Idenitities from trigonometry

Given a triangle of sides a, b and c and angles A, B, C and circumscribed by a circle of radius R.



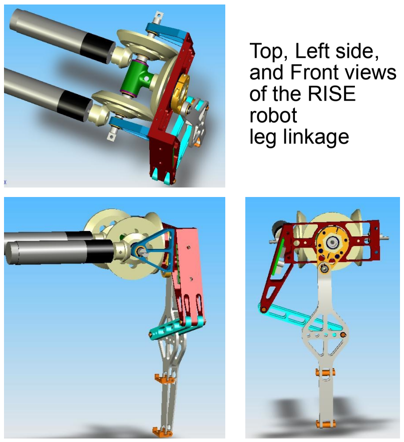
- The law of sines: $a/\sin A = b/\sin B = c/\sin C = 2R$
- The law of cosines: $a^2 + b^2 - 2ab\cos C = c^2$

Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$

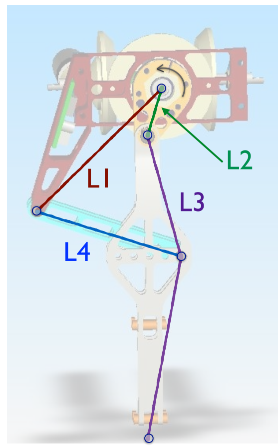
The Loop Equation:

Let each link of a 4-bar mechanism be considered as a vector, $\mathbf{R}_i$, from its starting joint to the next joint. Then $\mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_4 = \mathbf{0}$. (This is a vector equation so it results in two equations in x and y.)

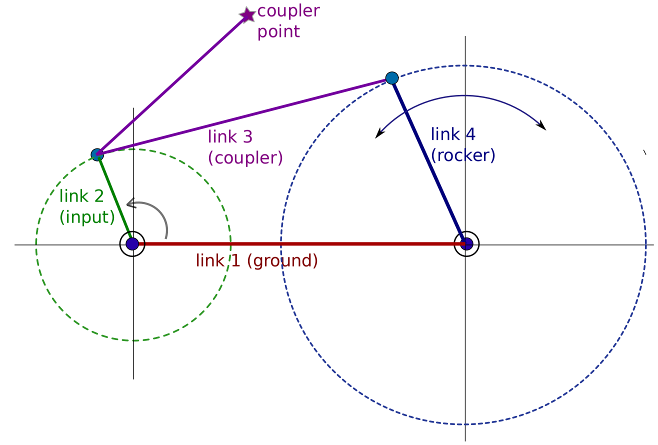
ME112: Four Bar Linkage Analysis



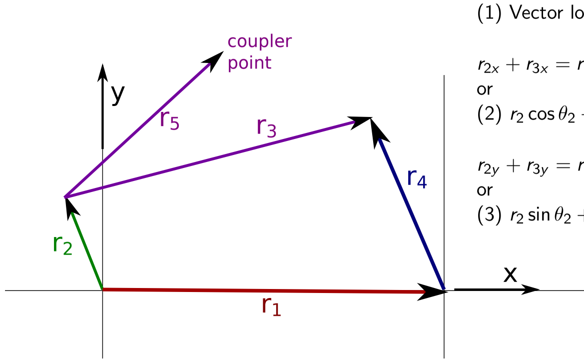
Top, Left side, and Front views of the RISE robot leg linkage



ME112: Four Bar Linkage Analysis



ME112: Four Bar Linkage Analysis

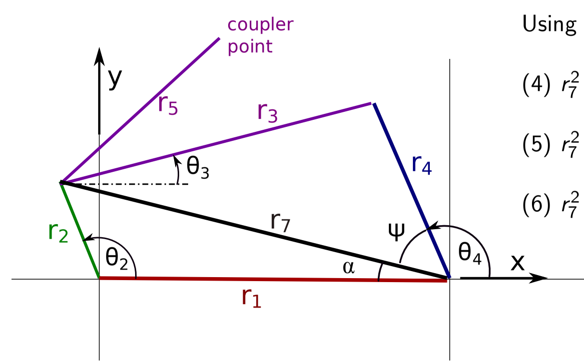


(1) Vector loop: $\vec{r_2} + \vec{r_3} = \vec{r_1} + \vec{r_4}$

$r_{2x} + r_{3x} = r_{1x} + r_{4x}$
or
(2) $r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_4 \cos\theta_4 - r_1 = 0$

$r_{2y} + r_{3y} = r_{1y} + r_{4y}$
or
(3) $r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_4 \sin\theta_4 = 0$

ME112: Four Bar Linkage Analysis



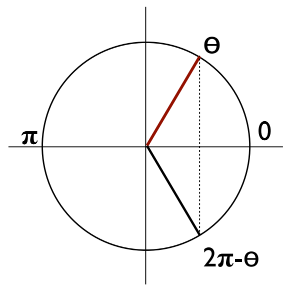
Using the law of cosines:

(4) $r_7^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta_2$

(5) $r_7^2 + r_4^2 - r_3^2 = 2r_1r_2\cos\psi$

(6) $r_7^2 + r_1^2 - r_2^2 = 2r_1r_7\cos\alpha$

ME112: Four Bar Linkage Analysis



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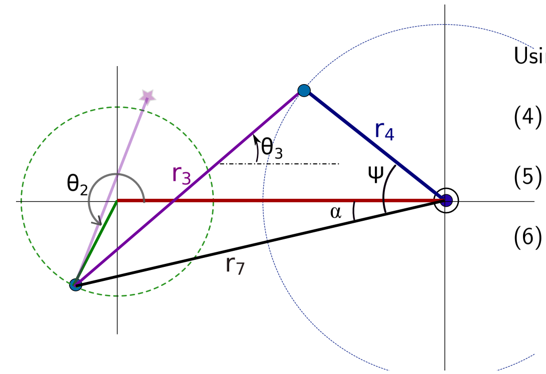
(6) $r_7^2 + r_1^2 - r_2^2 = 2r_1r_7\cos\alpha$

Two possible solutions for arccos( )

ME112: Four Bar Linkage Analysis



Using the law of cosines:

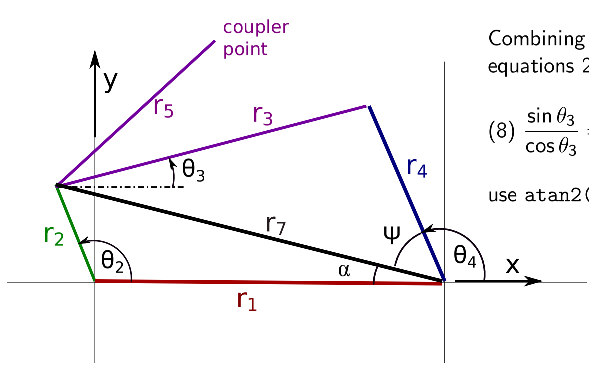
(4) $r_7^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta_2$

(5) $r_7^2 + r_4^2 - r_3^2 = 2r_1r_2\cos\psi$

(6) $r_7^2 + r_1^2 - r_2^2 = 2r_1r_7\cos\alpha$

alpha is sometimes negative

ME112: Four Bar Linkage Analysis



(7) $\theta_4 = \pi - \alpha - \psi$

Combining vector loop x and y equations 2 and 3:

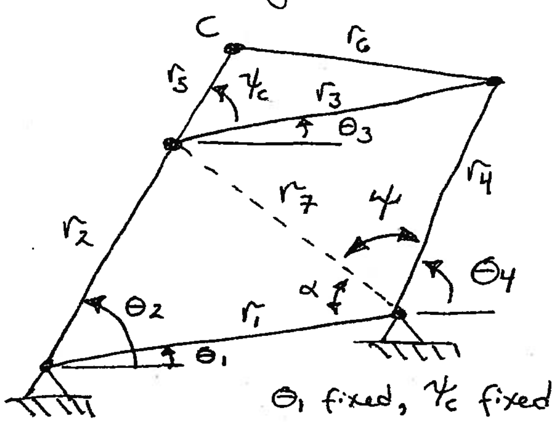
(8) $\dfrac{\sin\theta_3}{\cos\theta_3} = \dfrac{r_4\sin\theta_4 - r_2\sin\theta_2}{r_4\cos\theta_4 - r_2 cos\theta_2 + r_1}$

use `atan2(y, x)` to solve for θ_3

Summary:

- Depict linkage as a series of vectors: $\vec{r_1}, \vec{r_2}, \vec{r_3}, \vec{r_4}$.
- Add fictitious hypotenuse, and use Law of Cosines to solve for angles.
- Keep track of which quadrant $(0, \pi/2, \pi, 3\pi/2, 2\pi)$ angles are in.
- Using *tan half angle* identity, one can convert equations to a quadratic, for which the roots are the two inversions

Position Analysis of the four bar linkage



θ_1 fixed, ψ_c fixed

The loop equations for a four bar linkage:

x-direction: $r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_4 \cos\theta_4 - r_1 \cos\theta_1 = 0$

y-direction: $r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_4 \sin\theta_4 - r_1 \sin\theta_1 = 0$

If all link lengths are given and θ_1 is fixed, these can be solved once θ_2 is given since we have two equations and two unknowns. Unfortunately, we have two equations with nonlinear trig functions and the potential for zero solutions (if the links cannot be assembled to give that value of θ_2) or two solutions (one for each assembly or geometric inversion)

The loop equations will come in very handy for velocity analysis, but it is often more convenient to use the particular characteristics of the linkage to solve for positions or angles.

From the law of cosines:

$$r_7^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_2 - \theta_1)$$

Once r_7 is known, use the law of cosines to get ψ

$$r_3^2 = r_7^2 + r_4^2 - 2 r_4 r_7 \cos\psi$$

$$\Rightarrow \psi = \cos^{-1}\left(\frac{r_4^2 + r_7^2 - r_3^2}{2 r_4 r_7}\right) \qquad (\psi \text{ positive})$$

If $\theta_1 \le \theta_2 \le \theta_1 + \pi$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{r_1^2 + r_7^2 - r_2^2}{2 r_1 r_7}\right)$$

Otherwise $\Rightarrow \alpha = -\cos^{-1}\left(\frac{r_1^2 + r_7^2 - r_2^2}{2 r_1 r_7}\right)$

Now Θ_4 follows directly

$$\Theta_4 = \pi - (\alpha - \theta_1) - \psi$$

$$= \pi - \alpha + \theta_1 - \psi \quad \text{in the configuration shown}$$

For the other geometric inversion or assembly of the linkage

$$\Theta_4 = \pi - \alpha + \theta_1 + \psi$$

Once θ_2 and θ_4 are known, θ_3 can be calculated from the loop equations...

$$r_3 \cos\theta_3 = r_4 \cos\theta_4 + r_1 \cos\theta_1 - r_2 \cos\theta_2$$

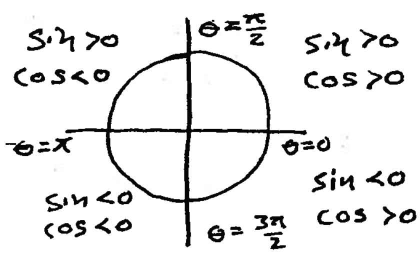
so if $\quad r_4 \cos\theta_4 + r_1 \cos\theta_1 - r_2 \cos\theta_2 > 0$
$\quad \Longrightarrow \cos\theta_3 > 0$

if $\quad r_4 \cos\theta_4 + r_1 \cos\theta_1 - r_2 \cos\theta_2 < 0$
$\quad \Longrightarrow \cos\theta_3 < 0$

Since $r_3 \sin\theta_3 = r_4 \sin\theta_4 + r_1 \sin\theta_1 - r_2 \sin\theta_2$

$$\theta_3^T = \sin^{-1}\left(\frac{r_4 \sin\theta_4 + r_1 \sin\theta_1 - r_2 \sin\theta_2}{r_3}\right)$$

This gives a value of θ_3^T such that $-\pi \le \theta_3^T \le \pi$
This corresponds to the value of θ_3 when $\cos\theta_3 > 0$



So if $\cos\theta_3 > 0$
$\Longrightarrow \theta_3 = \theta_3^T$

if $\cos\theta_3 < 0$
$\theta_3 = \pi - \theta_3^T$

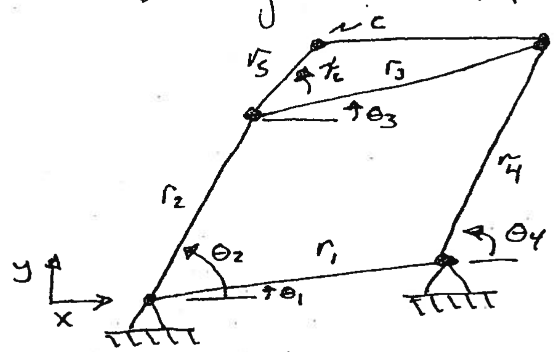
With $\theta_1 \rightarrow \theta_4$ known, the position analysis of the linkage is complete. Any point on the linkage can now be determined.

For example, the coupler point c is given by

$$x_c = r_2 \cos\theta_2 + r_3 \cos(\theta_3 + \psi_c)$$
$$y_c = r_2 \sin\theta_2 + r_3 \sin(\theta_3 + \psi_c)$$

Velocity Analysis of Mechanisms

Position analysis is often the most difficult part of analyzing a mechanism. Velocity analysis can be performed very systematically after the position analysis and concepts of power then yield certain forces without much work.



For the four-bar linkage, the loop equations and coupler point position are given by:

$$x_c = r_2 \cos\theta_2 + r_5 \cos(\theta_3 + \gamma_c)$$
$$y_c = r_2 \sin\theta_2 + r_5 \sin(\theta_3 + \gamma_c)$$

$$r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_4 \cos\theta_4 - r_1 \cos\theta_1 = 0$$
$$r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_4 \sin\theta_4 - r_1 \sin\theta_1 = 0$$

We can find velocities by simply differentiating:

$$-r_2 \sin\theta_2 \dot{\theta}_2 - r_3 \sin\theta_3 \dot{\theta}_3 + r_4 \sin\theta_4 \dot{\theta}_4 = 0$$
$$r_2 \cos\theta_2 \dot{\theta}_2 + r_3 \cos\theta_3 \dot{\theta}_3 - r_4 \cos\theta_4 \dot{\theta}_4 = 0$$

Once we have the position analysis solution, this is a linear set of equations we can rewrite as

$$A\omega_3 + B\omega_4 = C\omega_2 \qquad \omega_2 = \dot{\theta}_2 \qquad \omega_4 = \dot{\theta}_4$$
$$D\omega_3 + E\omega_4 = F\omega_2 \qquad \omega_3 = \dot{\theta}_3$$

where

$A = -r_3 \sin\theta_3$ $\qquad D = r_3 \cos\theta_3$
$B = r_4 \sin\theta_4$ $\qquad E = -r_4 \cos\theta_4$
$C = r_2 \sin\theta_2$ $\qquad F = -r_2 \cos\theta_2$

Solving gives

$$\frac{\omega_3}{\omega_2} = \frac{BF - EC}{DB - EA} \qquad \frac{\omega_4}{\omega_2} = \frac{AF - DC}{EA - DB}$$

Since this is a one degree of freedom mechanism we can only solve for ratios of velocities. If ω_2 is known, however, we can solve for the other velocities.

It is straightforward to solve for other velocities in terms of the input angular velocity as well.

To get the velocity of the coupler point, differentiate:

$$V_{x_c} = \dot{x}_c = -r_2 \sin\theta_2\, \omega_2 - r_5 \sin(\theta_3 + \gamma_c)(\omega_3 + \cancel{\dot{\gamma}_c}^{\,0})$$

$$= -r_2 \sin\theta_2\, \omega_2 - r_5 \sin(\theta_3 + \gamma_c)\, \omega_3$$

$$V_{y_c} = \dot{y}_c = r_2 \cos\theta_2\, \omega_2 + r_5 \cos(\theta_3 + \gamma_c)(\omega_3 + \cancel{\dot{\gamma}_c}^{\,0})$$

$$= r_2 \cos\theta_2\, \omega_2 + r_5 \cos(\theta_3 + \gamma_c)\, \omega_3$$

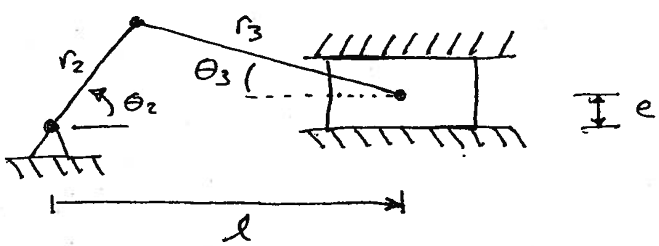
These can also be written as ratios of the input angular velocity ω_2

$$V_{x_c} = -r_2 \sin\theta_2 - r_5 \sin(\theta_3 + \gamma_c)\left(\frac{\omega_3}{\omega_2}\right)$$

$$V_{y_c} = r_2 \cos\theta_2 + r_5 \cos(\theta_3 + \gamma_c)\left(\frac{\omega_3}{\omega_2}\right)$$

where $\frac{\omega_3}{\omega_2}$ is given as before

The slider crank is analyzed similarly



Loop equations:

$$r_2 \sin\theta_2 - r_3 \sin\theta_3 - e = 0$$
$$l = r_2 \cos\theta_2 + r_3 \cos\theta_3$$

Differentiating:

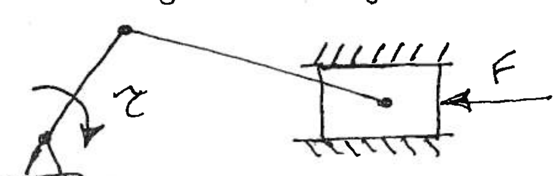
$$r_2 \cos\theta_2\, \omega_2 - r_3 \cos\theta_3\, \omega_3 = 0$$

$$\Longrightarrow \quad \frac{\omega_3}{\omega_2} = \frac{r_2 \cos\theta_2}{r_3 \cos\theta_2}$$

$$\dot{l} = -r_2 \sin\theta_2\, \omega_2 - r_3 \sin\theta_3\, \omega_3$$

$$\Longrightarrow \quad \frac{\dot{l}}{\omega_2} = -r_2 \sin\theta_2 - r_3 \sin\theta_3 \left(\frac{\omega_3}{\omega_2}\right)$$

If we neglect linkage mass, this leads to a quick force analysis



$$\tau \omega_2 = F \dot{l}$$

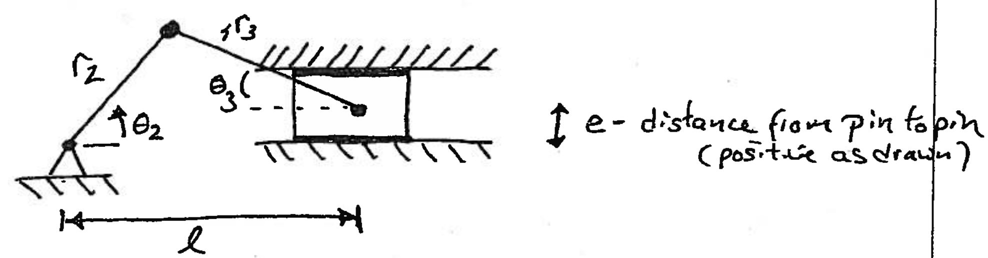
$$\frac{\tau}{F} = \frac{\dot{l}}{\omega} = -r_2 \sin\theta_2 - r_3 \sin\theta_3 \left(\frac{\omega_3}{\omega_2}\right)$$

Wow! That was easy!

The Amazing Slider Crank

The slider crank is another simple mechanism which – like the four-bar linkage – consists of four links. Instead of four pin joints it has three pin joints and a slider. (What is the simple mechanism with four links and two pins together with two sliders?)

A typical slider crank looks like:



Which you'll find (for instance) in an automotive engine.

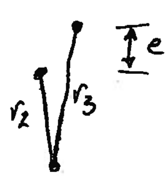
Just as with the four-bar linkage we can do a position and velocity analysis

Loop equations: $r_2 \sin\theta_2 - r_3 \sin\theta_3 - e = 0$

$$\ell = r_2 \cos\theta_2 + r_3 \cos\theta_3$$

If we fix θ_2, this gives 2 equations and 2 unknowns (θ_3 and ℓ). Do we have to worry about multiple solutions and things blowing up on us? We do, after all, have nonlinear equations involving trig. functions.

If we want to use this as a slider crank, the crank needs to be able to make a complete rotation, so



$$r_3 \geq r_2 + |e|$$

When $r_3 > r_2 + |e|$

$$-\frac{\pi}{2} < \theta_3 < \frac{\pi}{2}$$

So we can solve $\theta_3 = \sin^{-1}\left(\frac{r_2 \sin\theta_2 - e}{r_3}\right)$ without any hassle.

We can then get ℓ directly from the second loop equation. Wow position analysis of the slider crank is a breeze!

How about velocity analysis? Differentiating the loop equations gives:

$$r_2 \cos\theta_2\, \omega_2 - r_3 \cos\theta_3\, \omega_3 = 0$$

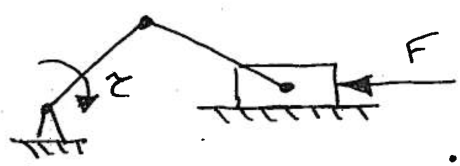
$$\Longrightarrow \quad \frac{\omega_3}{\omega_2} = \frac{r_2 \cos\theta_2}{r_3 \cos\theta_3}$$

$$\dot{l} = -r_2 \sin\theta_2\, \omega_2 - r_3 \sin\theta_3\, \omega_3$$

$$\frac{\dot{l}}{\omega_2} = -r_2 \sin\theta_2 - r_3 \sin\theta_3 \left(\frac{\omega_3}{\omega_2}\right)$$

That was easy too!

On to a virtual power analysis...



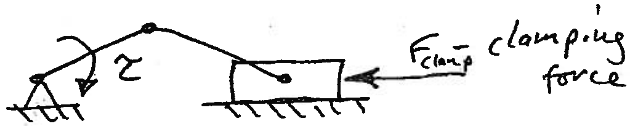
How do τ and F relate statically?

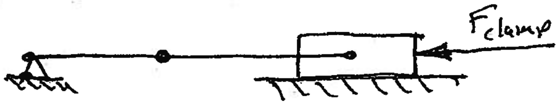
$$\tau \omega_2 = F \dot{l}$$

$$\frac{\tau}{F} = \frac{\dot{l}}{\omega} = -r_2 \sin\theta_2 - r_3 \sin\theta_3 \underbrace{\left(\frac{\omega_3}{\omega_2}\right)}_{= \frac{r_2 \cos\theta_2}{r_3 \cos\theta_3}}$$

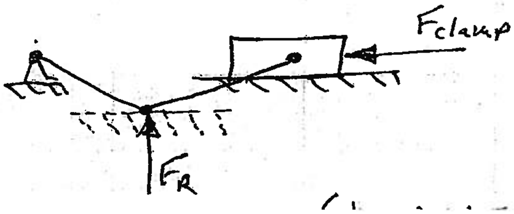
That's it! Of course other forces and mass can be added as well.

The slider crank has other applications besides the automotive engine. For instance, it can be used as a clamping device...



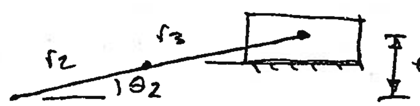


in theory, no torque is needed to sustain the force F

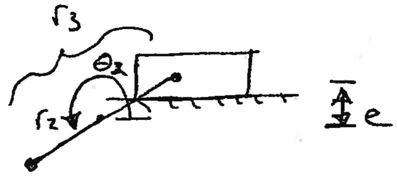


a more stable approach is to allow the mechanism to go over center and rest against a stop.

Another cool thing to do with a slider crank is to make a quick return mechanism. If the mechanism has an offset e, the extreme positions of motion are:



$$\theta_{2_{min}} = \sin^{-1}\left(\frac{e}{r_2+r_3}\right)$$



$$\theta_{2_{max}} = \sin^{-1}\left(\frac{e}{r_3-r_2}\right)+\pi$$

If the linkage moves at constant speed then the forward stroke takes a time t_f

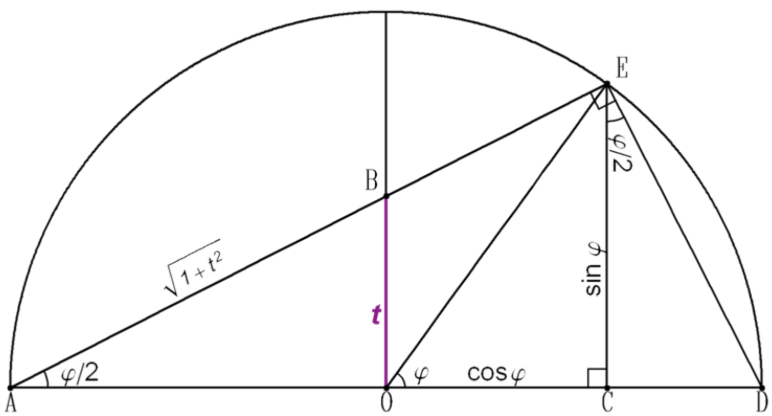
$$t_f = \frac{(\theta_{2max}-\theta_{2min})}{\omega_2}$$

the reverse stroke takes t_r

$$t_r = \frac{[2\pi-(\theta_{2max}-\theta_{2min}]}{\omega_2}$$

So the ratio $$\frac{t_f}{t_r} = \frac{\theta_{2max}-\theta_{2min}}{2\pi-(\theta_{2max}-\theta_{2min})}$$

Four Bar Solution using the Tangent Half Angle Method



$$t = \tan \tfrac{1}{2}\varphi = \frac{\sin\varphi}{1+\cos\varphi} = \frac{1-\cos\varphi}{\sin\varphi}.$$

source: http://en.wikipedia.org/wiki/Tangent_half-angle_formula

An inconvenience with the "straightforward" trigonometric analysis is that it can be tricky to catch the multiple possible solutions to the equations for various ranges of angles. For this reason, in the kinematics literature people often use complex exponential notation or the tangent half angle identities. The advantage to this approach is that we get a direct quadratic equation, the two solutions of which correspond exactly to the two possible inversions of the mechanism.

A good derivation is from M. Michael M. Stanisic', a kinematician at Notre Dame University (http://www.nd.edu/~stanisic/AME40423/):

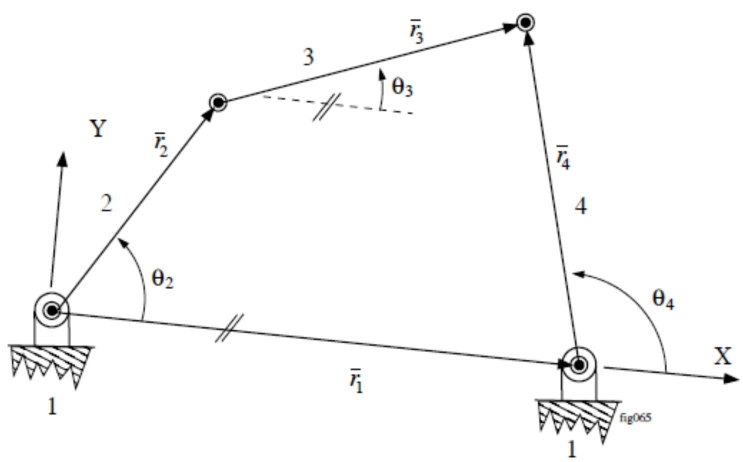


Figure 9.3: The four bar mechanism and its vector loop

Note that X is taken aligned with link 1, which simplifies the math slightly.
We get the normal loop equations:

$$r_2\cos\theta_2 + r_3\cos\theta_3 - r_4\cos\theta_4 - r_1 = 0 \tag{9.3}$$
$$r_2\sin\theta_2 + r_3\sin\theta_3 - r_4\sin\theta_4 = 0\,. \tag{9.4}$$

We can rearrange these a bit as follows:

$$
\begin{aligned}
r_3\cos\theta_3 &= -r_2\cos\theta_2 + r_4\cos\theta_4 + r_1 && (9.25) \\
r_3\sin\theta_3 &= -r_2\sin\theta_2 + r_4\sin\theta_4 \ , && (9.26)
\end{aligned}
$$

then square both sides of each equation and add the results together,

$$r_3^2 = r_2^2 + r_4^2 + r_1^2 - 2r_2r_4\cos\theta_2\cos\theta_4 + 2r_1r_4\cos\theta_4 - 2r_2r_1\cos\theta_2 - 2r_2r_4\sin\theta_2\sin\theta_4 \ . \quad (9.27)$$

This gives a single nonlinear equation in which the unknown is theta4. (Theta2 is the input.)

To make the solution more straightforward, it is useful to define some intermediate values: A, B, C. Then we introduce the tangent half-angle identity to solve for tan(theta4/2).

Begin by rewriting (9.27) as,

$$A\cos\theta_4 + B\sin\theta_4 = C \qquad (9.28)$$

where A, B and C are computed from the constants and the value of the input,

$$
\begin{aligned}
A &= 2r_4(r_1 - r_2\cos\theta_2) && (9.29) \\
B &= -2r_2r_4\sin\theta_2 && (9.30) \\
C &= r_3^2 - r_2^2 - r_4^2 - r_1^2 + 2r_2r_1\cos\theta_2 \ , && (9.31)
\end{aligned}
$$

Equation (9.28) can be solved in closed form for θ_4 by using the "tangent of the half angle formulas". In the case you are not familiar with these, they, along with the trigonometric "double angle formulas" are derived for you in the appendix. The tangent of the half angle formulas work like this.

Define u_4 as the tangent of the half angle of θ_4, i.e.

$$u_4 = \tan\left(\frac{\theta_4}{2}\right) \qquad (9.32)$$

then, as shown in the appendix,

$$\cos\theta_4 = \frac{1-u_4^2}{1+u_4^2} \quad \text{and} \quad \sin\theta_4 = \frac{2u_4}{1+u_4^2} \ . \qquad (9.33)$$

Substituting (9.33) into (9.28) yields a quadratic for u_4,

$$(C+A)u_4^2 - 2Bu_4 + (C-A) = 0 \ , \qquad (9.34)$$

which can be solved with the quadratic equation giving two values of u_4,

$$u_4 = \frac{B \pm [A^2 + B^2 - C^2]^{\frac{1}{2}}}{C+A} \ , \qquad (9.35)$$

The two solutions to u4 correspond to the two possible inversions of the mechanism. Having found u4, we use the equations above to back-solve for theta4 and finally theta3.

This is the method used in `HalfAngleMethod.m`, which will be posted on Coursework for running in Matlab or Octave and can be use to help check your own Matlab/Octave simulation code for your own mechanism.